MICHIGAN STATE UNIVERSITY

Name: ____________________
NetID: ____________________

Algebra Qualifying Exam I

August 2019

Instructions:

- Read each problem carefully.
- Write legibly and in complete sentences.
- Cross off anything you do not wish graded.
- This exam has 8 pages with 6 questions, for a total of 65 points. Make sure that all pages are included.
- You may not use books, notes, or electronic devices.
- You may ask proctors questions to clarify problems on the exam.
- Graded work should only be written in the provided spaces (not on scratch paper) unless otherwise approved by a proctor.
- You have 120 minutes to complete this exam.

Good luck!

1. (10 points) Let G be a group of order p^2, where p is a prime number. Prove that G is an abelian group.

2. (15 points) Let $\mathbf{F}_5$ be the finite field of order 5. The general linear group $GL_3(\mathbf{F}_5)$ consists of invertible 3×3 matrices over $\mathbf{F}_5$.

   (a) Determine the order of the group $GL_3(\mathbf{F}_5)$.

   (b) Prove that the unipotent subgroup

$$U = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; a, b, c \in \mathbf{F}_5 \right\}.$$

   is a 5-Sylow subgroup of $GL_3(\mathbf{F}_5)$.

(c) Recall that the normalizer of U is defined to be

$$B = \left\{ g \in GL_3(\mathbf{F}_5) \;\middle|\; gug^{-1} \in U \text{ for all } \; u \in U \right\}.$$

Prove that B is the subgroup of invertible upper triangular matrices.

3. (10 points) Consider the ring

$$\mathbb{Z}[\sqrt{-5}] = \left\{ a + b\sqrt{-5} \;\middle|\; a, b \in \mathbb{Z} \right\}.$$

Prove that $\mathbb{Z}[\sqrt{-5}]$ is not a unique factorization domain (UFD).

4. (10 points) Let R be a principal ideal domain (PID). Let S be a multiplicative subset of R such that $0 \notin S$. Prove that the localization $S^{-1}R$ is also a PID.

5. (10 points) Let $P_1 = x^7 + x^4 + x^3 + x^2 + x + 1$ and $P_2 = x^5 + x^4 + x^3 + x^2 + x + 1$. Determine the greatest common divisor of the polynomials P_1 and P_2 in the polynomial ring $\mathbb{Q}[x]$.

6. (10 points) Let $\mathbb{Z}$ be the ring of integers and let $\mathbb{Q}$ be the field of rational numbers. Is $\mathbb{Q}$ is projective $\mathbb{Z}$-module? Please justify your conclusion.